This week

1. Section 5.1: area estimating with finite sums
2. Section 5.2: limits of finite sums
3. Section 5.3: the definite integral
4. Section 5.4: the fundamental theorem of calculus

The integraph, an instrument for measuring integrals
The **Σ**-notation

We can write sums with the **Σ**-notation:

\[ \sum_{k=M}^{N} a_k = a_M + a_{M+1} + a_{M+2} + \cdots + a_{N-1} + a_N \]

- **Σ** is the Greek letter “S” (pronounced as 'sigma'), which refers to “Sum”.
- **k** is called the **index**.
- The index starts counting at **M** and stops counting at **N**.
- **a_k** is the **k-th term** of the sum, and is a formula containing **k**.
- If **N** < **M** then the sum is equal to 0 by definition.
- The index is a **dummy**.

\[ \sum_{k=3}^{6} a_k = \sum_{p=3}^{6} a_p = a_3 + a_4 + a_5 + a_6 \]

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The **Σ**-notation

\[ \sum_{k=1}^{12} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 12^2 \]
\[ = 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + 121 + 144 \]
\[ = 650. \]
Examples

\[ \sum_{k=1}^{4} (-1)^{k+1} = \]

\[ \sum_{k=1}^{2} \frac{k}{k+1} = \]

Arithmetic series

Theorem

The sum of the first \( n \) positive integers is equal to \( \frac{n(n+1)}{2} \).

- Define \( S_n \) as the sum of the first \( n \) positive integers:
  \[ S_n = 1 + 2 + \cdots + (n - 1) + n. \]
  
  - with \( \Sigma \)-notation:
    \[ S_n = \sum_{k=1}^{n} k. \]
  
  - Write out the terms in \( S_n \) twice:
    \[ S_n = 1 + 2 + \cdots + (n - 1) + n \]
    \[ S_n = n + (n - 1) + \cdots + 2 + 1 \]
Rules

- **Sum- and difference rule:**
  \[ \sum_{k=M}^{N} (a_k + b_k) = \sum_{k=M}^{N} a_k + \sum_{k=M}^{N} b_k, \quad \text{and} \quad \sum_{k=M}^{N} (a_k - b_k) = \sum_{k=M}^{N} a_k - \sum_{k=M}^{N} b_k. \]

- **Constant multiple rule:**
  \[ \sum_{k=M}^{N} c \ a_k = c \sum_{k=M}^{N} a_k. \]

- **Constant value rule:**
  \[ \sum_{k=M}^{N} c = (N - M + 1) c. \]

- **Splitting rule:**
  \[ \sum_{k=M}^{N} a_k = \sum_{k=M}^{P} a_k + \sum_{k=P+1}^{N} a_k. \]

Example

- Define \( T_n \) as the sum of the first \( n \) odd integers:
  \[ T_n = 1 + 3 + \cdots + (2n - 1). \]

- Notice that
  \[ T_n + (2 + 4 + \cdots + 2n) = 1 + 2 + 3 + \cdots + (2n - 1) + 2n = \frac{2n(2n + 1)}{2} = n(2n + 1) = 2n^2 + n. \]

- Furthermore
  \[ 2 + 4 + \cdots + 2n = \sum_{k=1}^{n} 2k = 2 \sum_{k=1}^{n} k = 2 \cdot \frac{n(n + 1)}{2} = n(n + 1) = n^2 + n. \]

- Therefore
  \[ T_n = (2n^2 + n) - (n^2 + n) = n^2. \]
Example

The sum of the first $n$ odd integers is equal to $n^2$:

\[
\begin{array}{cccccccc}
1 & 1 \\
2 & 3 \\
3 & 5 \\
4 & 7 \\
5 & 9 \\
6 & 11 \\
\vdots & \\
\end{array}
\]

Partitions

2.1

Definition

A partition of the interval $[a, b]$ in $n$ subintervals is a sequence $x_0, x_1, \ldots, x_n$ constructed as follows:

(i) $\Delta x = \frac{b - a}{n}$

(ii) $x_k = a + k\Delta x$ \hspace{1cm} ($k = 0, 1, \ldots, n$)

- Note that
  
  $x_0 = a$, \hspace{1cm} $x_n = b$, \hspace{1cm} $x_k - x_{k-1} = \Delta x$.

- The number $\Delta x$ is called the mesh of the partition.
Integrals as limits of Riemann sums

Approximate the area of the triangle with vertices \((0,0), (1,0)\) and \((1,1)\) with a Riemann sum.

- Define the partition \(x_k = k \Delta x = \frac{k}{n}\) with \(\Delta x = \frac{1}{n}\).
- The Riemann sum of \(f(x) = x\) is
  \[
  \sum_{k=1}^{n} x_k \Delta x = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n}.
  \]
Integrals as limits of Riemann sums

3.2

Evaluate the Riemann sum:

\[ \sum_{k=1}^{n} x_k \Delta x = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \]

If we let \( n \) approach infinity then

\[ \lim_{n \to \infty} \sum_{k=1}^{n} x_k \Delta x = \]

For a positive function, a Riemann sum can be regarded as the approximation of the surface area of the region \( R \) bounded by the graph of \( f \), the \( x \) axis, and the lines \( x = a \) and \( x = b \).

Definition

The definite integral of \( f \) over the interval \([a, b]\) is defined as

\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \left( \sum_{k=1}^{n} f(x_k) \cdot \Delta x \right) \]

A definite integral can be regarded as the area of the region \( R \).
Laws of integration

- The variable in the integral is a dummy:
  \[ \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(u) \, du \]

- Linearity:
  \[ \int_{a}^{b} \alpha f(x) + \beta g(x) \, dx = \alpha \int_{a}^{b} f(x) \, dx + \beta \int_{a}^{b} g(x) \, dx \]

- Additivity:
  \[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \]

- Interchanging the upper and lower limit gives a minus sign:
  \[ \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \]

Constant functions

\[ \int_{a}^{b} c \, dx = c(b - a) \]

Notice that the Riemann sum of any partition is
\[ \sum_{k=1}^{n} c \Delta x = n \cdot c \Delta x = c \frac{b - a}{n} = c(b - a). \]
Laws of integration

\[ \int_a^b x \, dx = \frac{1}{2} b^2 - \frac{1}{2} a^2 \]

- Note that \( \frac{1}{2} b^2 - \frac{1}{2} a^2 = \frac{1}{2} (b + a)(b - a) = \text{Area}(R) \).
- The regions \( T \) and \( R \) have the same area.

Displacement and velocity

- Differentiate displacement to compute velocity:
  \[ v(t) = s'(t) \]
- The displacement can be computed from the velocity by integrating:
  \[ s(t) = \lim_{n \to \infty} \sum_{k=1}^{n} v(t_k) \Delta t = \int_0^t v(\tau) \, d\tau \]

The integral \( \int_0^t v(\tau) \, d\tau \) is a function \( s(t) \) whose derivative is \( v \).
**Antiderivatives**

**Definition**

We call a function $F$ an **antiderivative** for $f$ if $F'(x) = f(x)$.

- Antiderivatives are not unique. If $F$ is an antiderivative for $f$, then so is $F(x) + C$ for any constant $C$:
  \[
  \frac{d}{dx}(F(x) + C) = F'(x) = f(x).
  \]

**Theorem**

Let $(x_0, y_0)$ be a point in the plane. Then there is a unique antiderivative $F$ of $f$ for which $F(x_0) = y_0$.

**Example**

- Let $f(x) = e^x + 1$, then $F(x) = e^x + x$ is an antiderivative of $f$.
- For arbitrary $C$ the function
  \[
  F_c(x) = e^x + x + C
  \]
  is also an antiderivative of $f$.
- There is only one antiderivative of $f$ for which $F(0) = 4$:
  \[
  F(x) = e^x + x + 3.
  \]
- The correct value for $C$ is found by solving the equation $F_c(0) = 4$:
  \[
  4 = F_c(0) = e^0 + 0 + C = 1 + C,
  \]
  hence $C = 3$. 
The inverse of differentiation

The Fundamental Theorem of Calculus

1. Define the function
   \[ F(x) = \int_a^x f(t) \, dt, \]
   then \( F \) is an antiderivative for \( f \), in other words: \( F'(x) = f(x) \).

2. If \( F \) is an antiderivative for \( f \) then
   \[ \int_a^b f(t) \, dt = F(b) - F(a). \]

- Notation: \( F(b) - F(a) = \left[ F(x) \right]_a^b = F(x) \bigg|_a^b \).
- The function \( F(x) = \int_a^x f(t) \, dt \) also satisfies \( F(a) = 0 \), so \( F \) is the unique antiderivative of \( f \) for which \( F(a) = 0 \).

Define \( F(x) = \int_a^x f(t) \, dt \), then

\[
F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt = f(x).
\]
Integrals with $e^x$

The fundamental theorem of Calculus:

$$\int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a) \quad \text{where } F' = f.$$ 

- $\int_{\ln 2}^0 e^x \, dx = \quad \text{(answer)}$

Integrals with $\sin$ and $\cos$

The fundamental theorem of Calculus:

$$\int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a) \quad \text{where } F' = f.$$ 

- $\int_0^{\pi/2} \cos(x) \, dx = \quad \text{(answer)}$
- $\int_{\pi}^{2\pi} \sin(x) \, dx = \quad \text{(answer)}$
Power functions

- Notice that for arbitrary real $\alpha$ we have
  \[ \frac{d}{dx}(x^{\alpha+1}) = (\alpha + 1)x^\alpha. \]

- Hence, if $\alpha \neq -1$:
  \[ \frac{d}{dx}\left(\frac{1}{\alpha+1}x^{\alpha+1}\right) = x^\alpha. \]

The antiderivative of $x^\alpha$ is: $\frac{1}{\alpha+1}x^{\alpha+1} + C$ if $\alpha \neq -1$.

The antiderivative of $x^{-1} = \frac{1}{x}$ is: $\ln|x| + C$.

See lecture 5

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Integrals with powers of $x$

The fundamental theorem of Calculus:

\[ \int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a) \quad \text{where } F' = f. \]

- $\int_0^2 x^3 \, dx =$
\int_{0}^{1} 2x^3 - 2x + 1 \, dx =